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An algorithm for semi-infinite transportation problems

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Abstract

In this paper we consider a class of semi-infinite transportation problems. We develop an algorithm for this class of semi-infinite transportation problems. The algorithm is a primal dual method which is a generalization of the classical algorithm for finite transportation problems. The most important aspect of our paper is that we can prove the convergence result for the algorithm. Finally, we implement some examples to illustrate our algorithm.

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Keywords: Semi-infinite transportation problem; Algorithm; Optimal solution

1. Introduction

In this paper we mainly consider the algorithm for the continuous transportation problem (CTP). A large number of papers [1,2,4–11,13] have appeared in the literature on this problem. They have mostly been concerned with the duality theory of the CTP and the existence of optimal solutions for such a problem. Only a few of papers [1,2,10,11,13] discuss the algorithms for some special classes of CTPs.

The classical transportation problem (TP) is a linear program posed in R^{mn} as follows:

$$\begin{aligned}
\text{minimize}\quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}\\
\text{subject to}\quad & \sum_{j=1}^{n} x_{ij}=a_i,\quad i=1,2,\ldots,m,\\
& \sum_{i=1}^{m} x_{ij}=b_j,\quad j=1,2,\ldots,n,\\
& x_{ij}\geqslant 0,\quad i=1,2,\ldots,m,\quad j=1,2,\ldots,n.
\end{aligned}$$

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The decision variables x_{ij} represent the amount shipped from source i to destination j. The demand at destination j is b_j and the supply at source i is a_i. In order to make the problem consistent, we need $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. The CTP is a continuous version of the classical TP. Now we formulate the CTP as follows:

$$\begin{aligned} &\text{minimize} && \int_{X\times Y} c(x, y)\,\mathrm{d}\rho(x, y) \\ &\text{subject to} && \rho(S\times Y) = \mu_1(S) \quad \text{for any measurable set } S \text{ in } X, \\ &&& \rho(X\times S) = \mu_2(S) \quad \text{for any measurable set } S \text{ in } Y, \\ &&& \rho \geqslant 0. \end{aligned}$$

Here ρ, μ_1, and μ_2 are nonnegative regular Borel measures and c is a continuous function. X and Y are compact Hausdorff spaces with $\mu_1(X) = \mu_2(Y)$. There are numerous applications in the field of the CTP. For example, in Civil Engineering, we have the following problem: Let $\mu_1(x_i)$ be the amount of units of soil in n distinct locations $x_i (i = 1, \ldots, n)$. We wish to transport the soil from the above n distinct locations to a certain highway which is under construction. The amount of soil that is needed for every given length E in the highway is $\mu_2(E) = \int_E h(y)\,\mathrm{d}y$, where $h(y)$ is a given function. The question is if the cost needed to move every unit amount of soil from location x_i to the yth position of the highway is $c(x_i, y)$, then what is the minimum cost that needed for the transportation of the soil? This TP is formulated by $X = \{x_1, \ldots, x_n\}$ and Y, where Y is a closed interval in R and μ_1, μ_2 are defined as above with $\mu_1(X) = \mu_2(Y)$. In this paper we will develop an algorithm for solving this kind of problem. It is well known that the dual problem for the continuous transportation problem (DCTP) has the following form:

$$\begin{aligned} &\text{maximize} && \int_X r(x)\,\mathrm{d}\mu_1(x) + \int_Y s(y)\,\mathrm{d}\mu_2(y) \\ &\text{subject to} && r(x) + s(y) \leqslant c(x, y) \quad \text{for each } (x, y) \in X\times Y, \end{aligned}$$

where r and s are continuous functions on X and Y, respectively. From the duality theory of CTP, we know if ρ is feasible for CTP and (r, s) is feasible for DCTP, then $\int_{X\times Y} c(x, y)\,\mathrm{d}\rho(x, y) \geqslant \int_X r(x)\,\mathrm{d}\mu_1(x) + \int_Y s(y)\,\mathrm{d}\mu_2(y)$. We denote $v(\text{CTP})$ and $v(\text{DCTP})$ as the optimal values for CTP and DCTP, respectively. Kretschmer [7] has proved a strong duality result for CTP and DCTP. Essentially this states that $v(\text{CTP}) = v(\text{DCTP})$. Anderson and Philpott [2] developed an algorithm to solve a simple version of CTP with μ_1 and μ_2, which are both Lebesgue measures. Anderson and Nash [1] and Lewis [10] discussed the semi-finite TP, which has some relation to the version of our problem. From Wu [13], we know the CTP has an optimal solution at an extreme point of the feasible region. Anderson and Nash [1] proved that there is an optimal solution for DCTP.

In this paper we consider the case $X = \{x_1, \ldots, x_n\}$, $Y = [0, 1]$, and μ_1 and μ_2 are defined as above. We intend to modify the algorithm in Anderson and Nash [1] and give a convergence proof for this algorithm to solve CTP. We characterize the extreme points structure for CTP and DCTP in Section 2. In Section 3 we use the idea from Anderson and Nash [1] to give an algorithm for CTP and prove the convergence result for this algorithm. Finally, we implement some numerical examples to illustrate our algorithm in Section 4.

2. Extreme points

In this section we discuss extreme points for CTP. From now we let $X = \{x_1, \ldots, x_n\}$, $Y = [0, 1]$, μ_1 be a discrete measure concentrated on X, and μ_2 be an absolutely continuous measure with respect to Lebesgue measure.

From Wu [13], we have the following Theorems 2.1 and 2.2.

Theorem 2.1. *Let ρ be a regular Borel measure with $\operatorname{supp}(\rho) = \bigcup_{i=1}^{n}\{x_i \times S_i\}$, where S_i is a finite union of intervals of R for $i = 1, 2, \ldots, n$ and the length of $S_i \cap S_j$ is equal to 0 for $i \neq j$. If ρ is feasible for CTP, then ρ is an extreme point of CTP.*

If ρ is feasible for CTP, then $\operatorname{supp}(\rho) \subseteq \bigcup_{i=1}^{n}\{x_i \times Y\}$. Now we assume that $\operatorname{supp}(\rho) = \bigcup_{i=1}^{n}\{x_i \times S_i\}$, where S_i is a finite union of intervals of R for $i = 1, \ldots, n$ and $\rho(x_i \times E) = \int_E h(y)\,\mathrm{d}y$ for every measurable set $E \subset S_i$, where $\inf_{y\in S_i} h(y) > 0$ for $i = 1, 2, \ldots, n$.

Theorem 2.2. *Let ρ be defined as above and feasible for CTP. If ρ is an extreme point of CTP, then the length of $S_i \cap S_j$ is equal to 0 for $i \neq j$.*

Now we give a condition for an extreme point to be an optimal solution for CTP. Let ρ be an extreme point defined as in Theorem 2.1, and there exists r_i^*, $i = 1, 2, \ldots, n$, such that $s^*(y)$ be defined as follows:

$$s^*(y) = c(x_i, y) - r_i^* \quad \text{for every } y \in S_i, \quad i = 1, 2, \ldots, n.$$

From Wu [13], we know if $s^*(y)$ is continuous on Y and $r_i^* + s^*(y) \leqslant c(x_i, y)$ for $i = 1, 2, \ldots, n$, $y \in Y$, then ρ is an optimal solution for CTP. Now we discuss the conditions of a feasible solution of DCTP to be an extreme point.

Let $s(y) = \min_{1 \leqslant i \leqslant n}\{c(x_i, y) - r_i\}$ for $y \in Y$. Then we have $s(y) + r_i \leqslant c(x_i, y)$ for $y \in Y$ and $i = 1, 2, \ldots, n$. Hence, (r, s) is a feasible solution for DCTP. But this (r, s) may not be an extreme point of DCTP. We assume that there exists a $(k, k') \neq (0, 0)$ such that $(r - k, s + k')$ and $(r + k, s - k')$ are feasible solutions for DCTP. Since $(r, s) = (r - k, s + k')/2 + (r + k, s - k')/2$, it follows that (r, s) is not an extreme point of DCTP. In the following theorem we show that under some conditions (r, s) may be an extreme point.

Theorem 2.3. *Let $r_1 = 0$ and (r, s) be a feasible solution for* DCTP. *Then (r, s) is an extreme point of* DCTP *if and only if $s(y) = \min_{1 \leqslant i \leqslant n}\{c(x_i, y) - r_i\}$ for $y \in Y$.*

Proof. This proof follows from Anderson and Nash [1]. □

3. Algorithm

In this section we consider the case that $\mu_1(x_1), \mu_1(x_2), \ldots, \mu_1(x_n)$ are not necessarily all equal and μ_2 is an absolutely continuous measure with respect to Lebesgue measure with the defining function $h(y) \geqslant 0$ on Y. For any given $r_1, r_2, \ldots, r_n$, we define $s(y) = \min_{1 \leqslant i \leqslant n}\{c(x_i, y) - r_i\}$ for $y \in Y$, and $S_i = \{y \in Y | s(y) = c(x_i, y) - r_i\}$ for $i = 1, 2, \ldots, n$. From hereafter, we assume that S_i is a finite union of intervals of R for $i = 1, 2, \ldots, n$.

Theorem 3.1. *Let $f_i : [0, 1] \to R$ be continuous functions for $i = 1, 2, \ldots, n$. We define $s_n(y) = \min_{1 \leqslant i \leqslant n}\{f_i(y)\}$ for $y \in [0, 1]$. Then $s_n(y)$ is a continuous function on* $[0, 1]$.

Proof. It follows from Royden [12]. □

From Theorem 3.1, we know that $s(y)$ is continuous on Y. By the definition of $s(y)$, we have $r_i + s(y) \leqslant c(x_i, y)$ for each $y \in Y$ and $i = 1, 2, \ldots, n$. Moreover, $r_i + s(y) = c(x_i, y)$ for $y \in S_i$. If we assume that $\mu_2(S_i) = \mu_1(x_i)$ for $i = 1, 2, \ldots, n$, then we can construct a feasible solution ρ for CTP with $\mathrm{supp}(\rho) = \bigcup_{i=1}^n \{x_i \times S_i\}$. It follows from a theorem in Wu [13] that ρ is an optimal solution for CTP. Now we develop an algorithm for CTP as follows:

Algorithm for semi-infinite transportation problem:

Step 0: Give $(r_1^l, r_2^l, \ldots, r_n^l) = (0, 0, \ldots, 0)$ and $l = 0$.

Step 1: Define $s^l(y) = \min_{1 \leqslant i \leqslant n}\{c(x_i, y) - r_i^l\}$ for $y \in Y$ and $S_i^l = \{y \in Y | s^l(y) = c(x_i, y) - r_i^l\}$ for $i = 1, 2, \ldots, n$. Go to step 3.

Step 2: We choose a suitable new $r_{i(l)}^l$ such that the defining function $s^l(y) = \min_{1 \leqslant i \leqslant n}\{c(x_i, y) - r_i^l\}$ for $y \in Y$ and $S_i^l = \{y \in Y | s(y) = c(x_i, y) - r_i^l\}$ satisfy $\mu_2(S_{i(l)}^l) = \mu_1(x_{i(l)})$.

Step 3: Compute $\mu_2(S_i^l)$ for $i = 1, 2, \ldots, n$.

Step 4: If $\mu_2(S_i^l) = \mu_1(x_i)$ for $i = 1, 2, \ldots, n$, then we stop the algorithm and there exists an optimal solution ρ for CTP with $\mathrm{supp}(\rho) = \bigcup_{i=1}^n \{x_i \times S_i\}$. Otherwise there exists $i(l) \in \{1, 2, \ldots, n\}$ such that $\max_{1 \leqslant i \leqslant n}\{\mu_2(S_i^l) - \mu_1(x_i)\} = \mu_2(S_{i(l)}^l) - \mu_1(x_{i(l)})$.

Step 5: Set $l \leftarrow l + 1$ and $r_i^l = r_i^{l-1}$ for $i \in \{1, 2, \ldots, n\}$ and $i \neq i(l)$. Go to step 2.

From now we assume that for each iteration l in the algorithm, the length of $S_i^l \cap S_j^l$ is equal to 0 for $i \neq j$.

Theorem 3.2. *For each l, we have $\bigcup_{i=1}^n S_i^l = Y$ and $\sum_{i=1}^n \mu_2(S_i^l) = \sum_{i=1}^n \mu_1(x_i)$.*

Proof. It is clear that $\bigcup_{i=1}^{n} S_i^l \subset Y$. For each fixed $y \in Y$, there exists $j \in \{1, 2, \ldots, n\}$ such that $s^l(y) = \min_{1 \leqslant i \leqslant n}\{c(x_i, y) - r_i^l\} = c(x_j, y) - r_j^l$. Hence, we have $y \in S_j^l$. This implies that $Y \subset \bigcup_{i=1}^{n} S_i^l$. Thus, we have $\bigcup_{i=1}^{n} S_i^l = Y$. Here we assume that the length of $S_i^l \cap S_j^l$ is equal to 0 for $i \neq j$. Thus, $\sum_{i=1}^{n} \mu_2(S_i^l) = \mu_2(Y) = \mu_1(X) = \sum_{i=1}^{n} \mu_1(x_i)$.We complete the proof. $\square$

From the algorithm and Theorem 3.2, we have $S_i^l \supseteq S_i^{l-1}$ for $i \neq i(l)$, $S_{i(l)}^l \subset S_{i(l)}^{l-1}$, $r_{i(l)}^l < r_{i(l)}^{l-1}$, $s^l(y) \geqslant s^{l-1}(y)$ for $y \in Y$. Moreover, $\{s^l(y) | l \in N\}$ is a sequence of increasing functions for every y in Y, and $\{r_i^l | l \in N\}$ is a decreasing sequence for $i = 1, 2, \ldots, n$.

Theorem 3.3. *Let $j \in \{1, 2, \ldots, n\}$ be fixed. If there exists $m(j) \in N$ such that $\mu_2(S_j^{m(j)}) \geqslant \mu_1(x_j)$, then $\mu_2(S_j^l) \geqslant \mu_1(x_j)$ for all $l \geqslant m(j)$.*

Proof. At the $m(j)$th iteration in the algorithm, we have either

$$\max_{1 \leqslant i \leqslant n} \{\mu_2(S_i^{m(j)}) - \mu_1(x_i)\} = \mu_2(S_j^{m(j)}) - \mu_1(x_j)$$

or

$$\max_{1 \leqslant i \leqslant n} \{\mu_2(S_i^{m(j)}) - \mu_1(x_i)\} > \mu_2(S_j^{m(j)}) - \mu_1(x_j).$$

First, we consider $\max_{1 \leqslant i \leqslant n}\{\mu_2(S_i^{m(j)}) - \mu_1(x_i)\} = \mu_2(S_j^{m(j)}) - \mu_1(x_j)$. This means that $\mu_2(S_j^{m(j)}) > \mu_1(x_j)$. At the $(m(j) + 1)$th iteration in the algorithm, we have $\mu_2(S_j^{m(j)+1}) = \mu_1(x_j)$. Now, we consider $\max_{1 \leqslant i \leqslant n}\{\mu_2(S_i^{m(j)}) - \mu_1(x_i)\} > \mu_2(S_j^{m(j)}) - \mu_1(x_j)$. At the $(m(j) + 1)$th iteration in the algorithm, we have $S_j^{m(j)+1} \supseteq S_j^{m(j)}$. This means that $\mu_2(S_j^{m(j)+1}) \geqslant \mu_2(S_j^{m(j)}) \geqslant \mu_1(x_j)$. Hence, $\mu_2(S_j^{m(j)+1}) \geqslant \mu_1(x_j)$ for all cases.

Similarly at the $(m(j) + 1)$th iteration in the algorithm, we also have either

$$\max_{1 \leqslant i \leqslant n} \{\mu_2(S_i^{m(j)+1}) - \mu_1(x_i)\} = \mu_2(S_j^{m(j)+1}) - \mu_1(x_j)$$

or

$$\max_{1 \leqslant i \leqslant n} \{\mu_2(S_i^{m(j)+1}) - \mu_1(x_i)\} > \mu_2(S_j^{m(j)+1}) - \mu_1(x_j).$$

If we consider the same argument as in the case of the $m(j)$th iteration for the $(m(j) + 1)$th iteration in the algorithm, then we can obtain $\mu_2(S_j^{m(j)+2}) \geqslant \mu_1(x_j)$ for all cases. For $l \geqslant m(j) + 2$ we repeat the same argument as above for lth iteration in the algorithm. Obviously we have $\mu_2(S_j^l) \geqslant \mu_1(x_j)$ for all $l \geqslant m(j)$. $\square$

Theorem 3.4. *Assume that the algorithm cannot stop at step 4 in a finite number of iterations. Then there exists $j \in \{1, 2, \ldots, n\}$ such that $\mu_2(S_j^l) < \mu_1(x_j)$ for all $l \in N$.*

Proof. We assume that there exists no $j \in \{1, 2, \ldots, n\}$ such that $\mu_2(S_j^l) < \mu_1(x_j)$ for all $l \in N$. Hence, for each $j \in \{1, 2, \ldots, n\}$, there exists $m(j) \in N$ such that $\mu_2(S_j^{m(j)}) \geqslant \mu_1(x_j)$. If we apply Theorem 3.3, then $\mu_2(S_j^l) \geqslant \mu_1(x_j)$ for all $l \geqslant m(j)$.

Let $m = \max\{m(1), m(2), \ldots, m(n)\}$. Then we have $\mu_2(S_j^l) \geqslant \mu_1(x_j)$ for all $l \geqslant m$, and $j = 1, 2, \ldots, n$. From Theorem 3.2, we have $\sum_{i=1}^{n} \mu_2(S_i^l) = \sum_{i=1}^{n} \mu_1(x_i)$ for all $l \in N$. Since $\mu_2(S_j^l) \geqslant \mu_1(x_j)$ for all $l \geqslant m$, and $j = 1, 2, \ldots, n$, we have $\mu_2(S_j^l) = \mu_1(x_j)$ for all $l \geqslant m$, and $j = 1, 2, \ldots, n$. This leads to a contradiction of our assumption. Thus there exists $j \in \{1, 2, \ldots, n\}$ such that $\mu_2(S_j^l) < \mu_1(x_j)$ for all $l \in N$. $\square$

Theorem 3.5. *Under the assumption that the algorithm cannot stop at step 4 at a finite number of iterations, there exists a $k \in \{1, 2, \ldots, n\}$ and an infinite subsequence $\{l_t | t \in N\}$ of $\{l | l \in N\}$ such that* $\max_{1 \leqslant i \leqslant n}\{\mu_2(S_i^{l_t}) - \mu_1(x_i)\} = \mu_2(S_k^{l_t}) - \mu_1(x_k)$ *for all $t \in N$.*

Proof. It is obvious. □

Theorem 3.6. *For each $i \in \{1, 2, \ldots, n\}$, we have that $\{r_i^l | l \in N\}$ is a convergent sequence.*

Proof. From the algorithm, we know that $\{r_i^l | l \in N\}$ is monotone decreasing for $i = 1, 2, \ldots, n$. Now we want to show that $\{r_i^l | l \in N\}$ is bounded below for $i = 1, 2, \ldots, n$. Assume to the contrary that there exists $k \in \{1, 2, \ldots, n\}$ such that $\{r_k^l | l \in N\}$ is not bounded below. By Theorem 3.4, there exists $j \in \{1, 2, \ldots, n\}$ such that $\mu_2(S_j^l) < \mu_1(x_j)$ for all $l \in N$. From step 5 of the algorithm, we know that for all $l \in N$, r_j^l are the same. Thus for all $l \in N$, the graphs of $c(x_j, y) - r_j^l$ are the same.

By the hypothesis, $\{r_k^l | l \in N\}$ is not bounded below, we have $\lim_{l\to\infty} r_k^l = -\infty$. Hence, by step 4 of the algorithm there exists an infinite subsequence $\{l_t | t \in N\}$ of $\{l | l \in N\}$ such that $\max_{1 \leqslant i \leqslant n}\{\mu_2(S_i^{l_t}) - \mu_1(x_i)\} = \mu_2(S_k^{l_t}) - \mu_1(x_k)$ for all $t \in N$ and $\lim_{t\to\infty} r_k^{l_t} = -\infty$. This implies that there exists $m(k) \in \{l_t | t \in N\}$ such that $\mu_2(S_k^{m(k)}) \geqslant \mu_1(x_k)$. Applying Theorem 3.3, we have $\mu_2(S_k^l) \geqslant \mu_1(x_k)$ for all $l \geqslant m(k)$.

Since $\lim_{l\to\infty} r_k^l = -\infty$, there exists $m_k \in N$ such that $c(x_k, y) - r_k^{m_k} > c(x_j, y) - r_j^1 = c(x_j, y) - r_j^{m(k)}$ for all $y \in Y$. Thus, we have $S_k^{m_k} = \emptyset$. Since $\{r_k^l | l \in N\}$ is monotone decreasing and $\{r_j^l | l \in N\}$ is a constant sequence, $c(x_k, y) - r_k^l \geqslant c(x_k, y) - r_k^{m_k} > c(x_j, y) - r_j^1 = c(x_j, y) - r_j^l$ for $y \in Y$ and $l \geqslant m_k$. This means that $S_k^l = \emptyset$ for all $l \geqslant m_k$. Thus $\mu_2(S_k^l) = 0$ for all $l \geqslant m_k$.

Let $m = \max\{m(k), m_k\}$. Then from the above results, we have both $\mu_2(S_k^l) \geqslant \mu_1(x_k) > 0$ and $\mu_2(S_k^l) = 0$ for all $l \geqslant m$. This leads to a contradiction. Therefore, $\{r_i^l | l \in N\}$ is bounded below for $i = 1, 2, \ldots, n$ and there exists $r_i \in R$ for $i = 1, 2, \ldots, n$ such that $\lim_{l\to\infty} r_i^l = r_i$. □

As in the proof of the above theorem, we define r_i as the limit point of $\{r_i^l | l \in N\}$ for $i = 1, 2, \ldots, n$. In order to prove the convergence result for the algorithm, we make one condition (c) to the family of functions $\{c(x_i, y) : y \in Y \text{ and } i = 1, 2, \ldots, n\}$ as follows:

Condition (c). For $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, there exists no $\alpha \in R$ and subinterval $[\beta, \eta] \subset Y$ such that $c(x_i, y) = \alpha + c(x_j, y)$ for all $y \in [\beta, \eta]$.

Now we denote

$$s(y) = \min_{1 \leqslant i \leqslant n} \{c(x_i, y) - r_i\} \quad \text{for each } y \text{ in } Y,$$

$$S_k = \{y \in Y | s(y) = c(x_k, y) - r_k\} \quad \text{for } k = 1, 2, \ldots, n. \tag{1}$$

Then we can easily prove $\bigcup_{k=1}^n S_k = Y$.

Theorem 3.7. $\lim_{l\to\infty} s^l(y) = s(y)$ *for each y in Y.*

Proof. It is clear from the fact that $\lim_{l\to\infty} r_i^l = r_i$ for every $i = 1, 2, \ldots, n$. □

Theorem 3.8. *Let $s(y)$ and S_i, $i = 1, 2, \ldots, n$, be defined as* (1). *Then the length of $S_j \cap S_k$ is equal to 0 for $j \neq k$ and $j, k \in \{1, 2, \ldots, n\}$.*

Proof. We assume that there exists $j, k \in \{1, 2, \ldots, n\}$ with $j \neq k$ such that the length of $S_j \cap S_k$ is not equal to 0. Thus, there exists a subinterval $[\beta, \eta]$ of Y such that $[\beta, \eta] \subset S_j \cap S_k$. This means that for all $y \in [\beta, \eta]$, $s(y) = \min_{1 \leqslant i \leqslant n}\{c(x_i, y) - r_i\} = c(x_j, y) - r_j = c(x_k, y) - r_k$. It follows that $c(x_j, y) = c(x_k, y) + (r_j - r_k)$ for all $y \in [\beta, \eta]$. This result contradicts to condition (c), the hypothesis on the family $\{c(x_i, y) : y \in Y \textit{ and } i = 1, 2, \ldots, n\}$. Thus the length of $S_j \cap S_k$ is equal to 0 for $j \neq k$ and $j, k \in \{1, 2, \ldots, n\}$. □

Theorem 3.9. *Let S_i for $i = 1, 2, \ldots, n$ be defined as* (1). *Then we have* $\lim_{l\to\infty} \mu_2(S_i^l) = \mu_2(S_i)$ *for $i = 1, 2, \ldots, n$.*

Proof. Let $E = \{y \in Y | y \text{ is a boundary point of } S_i^l \text{ or } S_i \text{ for } l \in N \text{ and } i = 1, 2, \ldots, n\}$. Then E is a countable set. The Lebesgue measure of E is 0. Since μ_2 is an absolutely continuous measure with respect to Lebesgue measure, we have

$\mu_2(E)=0$.

Now we want to prove for each $k \in \{1,2,\ldots,n\}$, if $y \in S_k$ and $y \notin E$, then there exists some $m \in N$ such that $y \in S_k^l$ for all $l \geqslant m$. We fix $k \in \{1,2,\ldots,n\}$. We assume $y \in S_k$ and $y \notin E$. By the definition of S_k and by the definition of E, we have

$$c(x_k, y) - r_k = \min_{1\leqslant i\leqslant n}\{c(x_i, y) - r_i\} < c(x_j, y) - r_j \tag{2}$$

for $j \in \{1,2,\ldots,n\}$ and $j \neq k$.

From the algorithm we know $\{r_j^l | l \in N\}$ is a decreasing sequence and from Theorem 3.6, we have $\lim_{l\to\infty} r_j^l = r_j$ for $j=1,2,\ldots,n$. Let $j \in \{1,2,\ldots,n\}$ and $j \neq k$. By (2), there exists $m(j) \in N$ such that $c(x_k, y) - r_k < c(x_j, y) - r_j^l$ for all $l \geqslant m(j)$. Let $m = \max\{m(1),\ldots,m(k-1),m(k+1),\ldots,m(n)\}$. Since $\{r_k^l | l \in N\}$ is a decreasing sequence and $\lim_{l\to\infty} r_k^l = r_k$, we have

$$c(x_k, y) - r_k^l \leqslant c(x_k, y) - r_k < c(x_j, y) - r_j^l \tag{3}$$

for all $l \geqslant m$ and $j \neq k$. From Theorem 3.7, we have for $l \geqslant m$, $s^l(y) \leqslant s(y) = c(x_k, y) - r_k$. From (3), we have, for all $l \geqslant m$, $c(x_k, y) - r_k^l < c(x_j, y) - r_j^l$ for $j \in \{1,2,\ldots,n\}$ and $j \neq k$. Applying the definition of $s^l(y)$, it follows that $s^l(y) = \min_{1\leqslant i\leqslant n}\{c(x_i, y) - r_i^l\} = c(x_k, y) - r_k^l$ for all $l \geqslant m$. This means that $y \in S_k^l$ for all $l \geqslant m$.

Next we want to prove that if $y \in Y, y \notin S_k$ and $y \notin E$, then there exists some $m \in N$ such that $y \notin S_k^l$ for $l \geqslant m$.

Since $\bigcup_{i=1}^n S_i = Y$, there exists $t \in \{1,2,\ldots,n\}$ such that $y \in S_t$ and $t \neq k$. By the same argument as above, there exists $m \in N$ such that $y \in S_t^l$ for all $l \geqslant m$. Since $y \notin E$ and $t \neq k$, we have $y \notin S_k^l$ for $l \geqslant m$.

Now we define

$$g_l(y) = \begin{cases} 1 & \text{if } y \in S_k^l, \\ 0 & \text{if } y \in Y - S_k^l \end{cases} \tag{4}$$

and

$$g(y) = \begin{cases} 1 & \text{if } y \in S_k, \\ 0 & \text{if } y \in Y - S_k. \end{cases} \tag{5}$$

From the above results, we have $\lim_{l\to\infty} g_l(y) = g(y)$ for almost every $y \in Y$ with respect to μ_2.

Since $|g_l(y)| \leqslant 1$ for all $y \in Y$ and $l \in N$, then we can apply the Lebesgue's dominated convergence theorem and get $\lim_{l\to\infty} \int_Y g_l(y)\,\mathrm{d}\mu_2(y) = \int_Y g(y)\,\mathrm{d}\mu_2(y)$.

Thus $\lim_{l\to\infty} \mu_2(S_k^l) = \mu_2(S_k)$. Similarly we can prove that $\lim_{l\to\infty} \mu_2(S_i^l) = \mu_2(S_i)$ for $i=1,2,\ldots,n$. $\square$

Theorem 3.10. *For $i=1,2,\ldots,n$, we have $\mu_2(S_i) = \mu_1(x_i)$.*

Proof. From Theorems 3.3 and 3.4, we have for each $i \in \{1,2,\ldots,n\}$ either exists $m(i) \in N$ such that $\mu_2(S_i^l) \geqslant \mu_1(x_i)$ for all $l \geqslant m(i)$ or $\mu_2(S_i^l) < \mu_1(x_i)$ for all $l \in N$. Let J be the collection of the index i which satisfies the property that there exists $m(i) \in N$ such that $\mu_2(S_i^l) \geqslant \mu_1(x_i)$ for all $l \geqslant m(i)$. Now we want to prove $\mu_2(S_i) = \mu_1(x_i)$ for all $i \in J$. From Theorem 3.9, we have $\lim_{l\to\infty} \mu_2(S_i^l) = \mu_2(S_i)$ for $i=1,2,\ldots,n$. Hence, we have $\mu_2(S_i) \geqslant \mu_1(x_i)$ for $i \in J$. We assume that there exists some $j \in J$ such that $\mu_2(S_j) > \mu_1(x_j)$. Let $\varepsilon = (\mu_2(S_j) - \mu_1(x_j))/2 > 0$. Since $\lim_{l\to\infty} \mu_2(S_j^l) = \mu_2(S_j)$, there exists $\bar{N} \in N$ such that $|\mu_2(S_j^l) - \mu_2(S_j)| < \varepsilon$ for all $l \geqslant \bar{N}$. It follows that $(\mu_1(x_j) - \mu_2(S_j))/2 < \mu_2(S_j^l) - \mu_2(S_j)$ for all $l \geqslant \bar{N}$. Thus $(\mu_1(x_j) + \mu_2(S_j))/2 - \mu_1(x_j) < \mu_2(S_j^l) - \mu_1(x_j)$ for all $l \geqslant \bar{N}$. Therefore we have

$$\varepsilon = \frac{\mu_2(S_j) - \mu_1(x_j)}{2} < \mu_2(S_j^l) - \mu_1(x_j) \tag{6}$$

for all $l \geqslant \bar{N}$.

From Theorem 3.5, we know there exists $k \in \{1,2,\ldots,n\}$ and an infinite subsequence $\{l_t | t \in N\}$ of $\{l | l \in N\}$ such that

$$\max_{1\leqslant i\leqslant n}\{\mu_2(S_i^{l_t}) - \mu_1(x_i)\} = \mu_2(S_k^{l_t}) - \mu_1(x_k) \tag{7}$$

for all $t \in N$. From the algorithm and (7), we know

$$\mu_2(S_k^{l_t+1}) = \mu_1(x_k) \quad \text{for all } t \in N. \tag{8}$$

From (6) we know $\max_{1 \leqslant i \leqslant n}\{\mu_2(S_i^l) - \mu_1(x_i)\} \geqslant \mu_2(S_j^l) - \mu_1(x_j) > \varepsilon$ for all $l \geqslant \bar{N}$.

Let $m_1 \in N$ satisfying if $t \geqslant m_1$, then $l_t \geqslant \bar{N}$. Thus we obtain that if $t \geqslant m_1$, then

$$\max_{1 \leqslant i \leqslant n}\{\mu_2(S_i^{l_t}) - \mu_1(x_i)\} = \mu_2(S_k^{l_t}) - \mu_1(x_k) \geqslant \mu_2(S_j^{l_t}) - \mu_1(x_j) > \varepsilon. \tag{9}$$

Thus, it follows from (8) and (9) that $\{\mu_2(S_k^l) | l \in N\}$ cannot converge. This is contradictory to the convergence of $\{\mu_2(S_k^l) | l \in N\}$. Hence, we have $\mu_2(S_i) = \mu_1(x_i)$ for all $i \in J$.

Let K be the collection of the index i which satisfies the property that $\mu_2(S_i^l) < \mu_1(x_i)$ for all $l \in N$. Since $\lim_{l \to \infty} \mu_2(S_i^l) = \mu_2(S_i)$ for $i = 1, 2, \ldots, n$, we have $\mu_2(S_i) \leqslant \mu_1(x_i)$ for all $i \in K$. Since $\mu_2(S_i) = \mu_1(x_i)$ for all $i \in J$ and $\sum_{i=1}^n \mu_2(S_i) = \sum_{i=1}^n \mu_1(x_i)$, we have $\mu_2(S_i) = \mu_1(x_i)$ for all $i \in K$. Hence $\mu_2(S_i) = \mu_1(x_i)$ for $i = 1, 2, \ldots, n$. □

Theorem 3.11. *If the algorithm does not stop in a finite number of iterations at step* 4, *then the sequence of* $(r^l, s^l(y))$ *converges to the optimal solution of* DCTP.

Proof. It follows from Theorems 3.6 to 3.10. □

Let (r^l, s^l) be obtained by lth iteration in our algorithm. By Theorem 3.11, the sequence (r^l, s^l) converges to the optimal solution of DCTP. Here, (r^l, s^l) can be viewed as an approximate solution of DCTP. We want to see how (r^l, s^l) approximates to the optimal solution of DCTP. We know (r^l, s^l) is feasible, we have

$$\begin{aligned}
&\int_X r^l(x)\,\mathrm{d}\mu_1(x) + \int_Y s^l(y)\,\mathrm{d}\mu_2(y) \\
&\quad = \sum_{i=1}^n r_i^l \mu_1(x_i) + \sum_{i=1}^n \int_{S_i^l} s^l(y)\,\mathrm{d}\mu_2(y) \\
&\quad \leqslant v(\mathrm{DCTP}) = v(\mathrm{CTP}).
\end{aligned}$$

Hence, we have

$$\sum_{i=1}^n r_i^l \mu_1(x_i) + \sum_{i=1}^n \int_{S_i^l} s^l(y)\,\mathrm{d}\mu_2(y) \leqslant v(\mathrm{CTP}). \tag{10}$$

From Theorem 3.2, we have $\sum_{i=1}^n \mu_2(S_i^l) = \sum_{i=1}^n \mu_1(x_i)$. Now, we assume that $I^l = \{i \in \{1, 2, \ldots, n\} | \mu_2(S_i^l) \geqslant \mu_1(x_i)\}$, $J^l = \{i \in \{1, 2, \ldots, n\} | \mu_2(S_i^l) < \mu_1(x_i)\}$. For $i \in I^l$, we decompose S_i^l as $S_i^l = A_i^l \cup B_i^l$ where A_i^l and B_i^l satisfies the following conditions:

(1) $\mu_2(A_i^l) = \mu_1(x_i)$,
(2) the length of $A_i^l \cap B_i^l$ is equal to 0,
(3) A_i^l and B_i^l are closed sets in R, A_i^l and B_i^l are finite union of intervals of R,
(4) the length of $A_i^l \cap A_j^l$ and $B_i^l \cap B_j^l$ are equal to 0 for $i \neq j$.

Define $B = \bigcup_{i \in I^l} B_i^l$. Then we have

$$\left(\bigcup_{i \in I^l} S_i^l\right) \cup \left(\bigcup_{i \in J^l} S_i^l\right) = \left(B \cup \left(\bigcup_{i \in I^l} A_i^l\right)\right) \cup \left(\bigcup_{i \in J^l} S_i^l\right) = Y.$$

We take any two members from $\{B_i^l | i \in I^l\}$, $\{A_i^l | i \in I^l\}$, or $\{S_i^l | i \in J^l\}$, then the length of the intersection of these two members is zero. Since $\sum_{i=1}^n \mu_2(S_i^l) = \sum_{i=1}^n \mu_1(x_i)$ and $\mu_2(A_i^l) = \mu_1(x_i)$ for $i \in I^l$, we decompose B suitably as $B = \bigcup_{j \in J^l} D_j^l$ where D_j^l for the index $j \in J^l$ satisfies the following conditions:

(1) D_j^l is a finite union of intervals of R, for $j \in J^l$,
(2) $\mu_2(S_j^l \cup D_j^l) = \mu_1(x_j)$, for $j \in J^l$,
(3) the length of $D_j^l \cap D_k^l$ is equal to 0 for $j \neq k$ and $j, k \in J^l$.

We define $A_j^l = S_j^l \cup D_j^l$ for $j \in J^l$. Then we have $\bigcup_{i=1}^n A_i^l = Y$ and the length of $A_i^l \cap A_j^l$ is equal to 0 for $i \neq j$. Thus, there exists a feasible solution ρ_l for CTP with $\mathrm{supp}(\rho_l) = \bigcup_{i=1}^n \{x_i \times A_i^l\}$. It is obvious that $\mu_2(B_i^l) = |\mu_2(S_i^l) - \mu_1(x_i)|$ for $i \in I^l$, $\mu_2(D_j^l) = |\mu_2(S_j^l) - \mu_1(x_j)|$ for $j \in J^l$.

Theorem 3.12. *Let A_i^l and B_i^l, $i = 1, 2, \ldots, n$, and ρ_l be defined as above. Define $M_i = \max_{0 \leqslant y \leqslant 1} |c(x_i, y)|$, $N_l = \max_{1 \leqslant i \leqslant n} |r_i^l|$, $M = \max\{M_1, M_2, \ldots, M_n, N_l\}$. Then we have $|\sum_{i=1}^n r_i^l \mu_1(x_i) + \sum_{i=1}^n \int_{S_i^l} s^l(y)\, \mathrm{d}\mu_2(y) - v(\mathrm{CTP})| \leqslant 2nM \max_{1 \leqslant i \leqslant n} \{|\mu_2(S_i^l) - \mu_1(x_i)|\}$.*

Proof. We have

$$
\begin{aligned}
&\int_{X \times Y} c(x, y) d\rho_l(x, y) \\
&= \sum_{i=1}^n \int_{\{x_i\} \times Y} c(x, y)\, \mathrm{d}\rho_l(x, y) \\
&= \sum_{i=1}^n \int_{X \times A_i^l} \chi_{\{x_i\}}(x) c(x, y)\, \mathrm{d}\rho_l(x, y) \\
&= \sum_{i=1}^n \int_{X \times Y} \chi_{A_i^l}(y) \chi_{\{x_i\}}(x) c(x, y)\, \mathrm{d}\rho_l(x, y) \\
&= \sum_{i=1}^n \int_{Y} \chi_{A_i^l}(y) c(x_i, y)\, \mathrm{d}\mu_2(y) \\
&= \sum_{i=1}^n \int_{A_i^l} c(x_i, y)\, \mathrm{d}\mu_2(y).
\end{aligned}
$$

Since ρ_l is a feasible solution of CTP, $v(\mathrm{CTP}) \leqslant \int_{X \times Y} c(x, y)\, \mathrm{d}\rho_l(x, y)$, which implies

$$
v(\mathrm{CTP}) \leqslant \sum_{i=1}^n \int_{A_i^l} c(x_i, y)\, \mathrm{d}\mu_2(y) \tag{11}
$$

From (10) and (11), we have

$$
\sum_{i=1}^n r_i^l \mu_1(x_i) + \sum_{i=1}^n \int_{S_i^l} s^l(y)\, \mathrm{d}\mu_2(y) \leqslant v(\mathrm{CTP}) \leqslant \sum_{i=1}^n \int_{A_i^l} c(x_i, y)\, \mathrm{d}\mu_2(y).
$$

Table 1

l	r_1^l	r_2^l	r_3^l
0	0	0	0
1	0	0	−0.25
2	0	−0.0417	−0.25
3	0	−0.0417	−0.2917
4	0	−0.08	−0.2917
5	0	−0.08	−0.33

Then we have

$$\begin{aligned}
&\left|\sum_{i=1}^{n} r_i^l \mu_1(x_i) + \sum_{i=1}^{n} \int_{S_i^l} s^l(y)\,\mathrm{d}\mu_2(y) - v(\mathrm{CTP})\right| \\
&\quad\leqslant \left|\sum_{i=1}^{n} r_i^l \mu_1(x_i) + \sum_{i=1}^{n} \int_{S_i^l} s^l(y)\,\mathrm{d}\mu_2(y) - \sum_{i=1}^{n} \int_{A_i^l} c(x_i, y)\,\mathrm{d}\mu_2(y)\right| \\
&\quad= \left|\sum_{i=1}^{n} r_i^l \mu_1(x_i) + \sum_{i=1}^{n} \int_{S_i^l} [c(x_i, y) - r_i^l]\,\mathrm{d}\mu_2(y) - \sum_{i=1}^{n} \int_{A_i^l} c(x_i, y)\,\mathrm{d}\mu_2(y)\right| \\
&\quad= \left|\sum_{i=1}^{n} r_i^l [\mu_1(x_i) - \mu_2(S_i^l)] + \sum_{i=1}^{n} \int_{S_i^l} c(x_i, y)\,\mathrm{d}\mu_2(y) - \sum_{i=1}^{n} \int_{A_i^l} c(x_i, y)\,\mathrm{d}\mu_2(y)\right| \\
&\quad\leqslant \sum_{i=1}^{n} |r_i^l||\mu_2(S_i^l) - \mu_1(x_i)| + \sum_{i\in I} \left|\int_{B_i^l} c(x_i, y)\,\mathrm{d}\mu_2(y)\right| + \sum_{i\in J} \left|\int_{D_i^l} c(x_i, y)\,\mathrm{d}\mu_2(y)\right| \\
&\quad\leqslant \sum_{i=1}^{n} N_l |\mu_2(S_i^l) - \mu_1(x_i)| + \sum_{i=1}^{n} M_i |\mu_2(S_i^l) - \mu_1(x_i)| \\
&\quad\leqslant nM \max_{1\leqslant i\leqslant n} \{|\mu_2(S_i^l) - \mu_1(x_i)|\} + nM \max_{1\leqslant i\leqslant n} \{|\mu_2(S_i^l) - \mu_1(x_i)|\} \\
&\quad= 2nM \max_{1\leqslant i\leqslant n} \{|\mu_2(S_i^l) - \mu_1(x_i)|\}. \qquad \square
\end{aligned}$$

From Theorem 3.12, we have an error bound for the difference between the objective value of the dual feasible solution in lth iteration of our algorithm and the optimal value of DCTP.

4. Implementation of the algorithm

In this section, we implement the algorithm given in Section 3 under the MATLAB(version 6.5) environment for solving some examples. The numerical results were done by using a Pentium 4 2.4 GHz personal computer. The stop criteria of step 4 of the algorithm at the lth iteration are $|\mu_2(S_i^l) - \mu_1(x_i)| < 10^{-5}$ for $i = 1, 2, \ldots, n$. In the following, we provide three examples to illustrate how our algorithm works to obtain the optimal solutions.

Example 4.1. Let x_1, x_2, x_3 be any three points in [0, 1]. We consider $c(x_1, y) = 2y$, $c(x_2, y) = y$, $c(x_3, y) = \frac{1}{2}$, for $y \in Y = [0, 1]$. Let $h(y) = 2y$, $0 \leqslant y \leqslant 1$. Then $\mu_2(Y) = \int_0^1 2y\,\mathrm{d}y = 1$. Let $\mu_1(x_1) = \frac{1}{16}$, $\mu_1(x_2) = \frac{1}{2}$, $\mu_1(x_3) = \frac{7}{16}$.

First, we list the results of the first five iterations are given in Tables 1–3.

At the 0th iteration, we let $(r_1^0, r_2^0, r_3^0) = (0, 0, 0)$. From the output results, we have $S_1^0 = \{0\}$, $S_2^0 = [0, \frac{1}{2}]$, $S_3^0 = [\frac{1}{2}, 1]$ and $\mu_2(S_1^0) = 0$, $\mu_2(S_2^0) = \frac{1}{4}$, $\mu_2(S_3^0) = \frac{3}{4}$. Moreover, we have $\max_{1\leqslant i\leqslant 3}\{\mu_2(S_i^0) - \mu_1(x_i)\} = \mu_2(S_3^0) - \mu_1(x_3)$.

Table 2

l	S_1^l	S_2^l	S_3^l
0	{0}	[0, 0.5]	[0.5, 1]
1	{0}	[0, 0.75]	[0.75, 1]
2	[0, 0.0147]	[0.0417, 0.7083]	[0.7083, 1]
3	[0, 0.0417]	[0.0417, 0.75]	[0.75, 1]
4	[0, 0.08]	[0.08, 0.7116]	[0.7116, 1]
5	[0, 0.08]	[0.08, 0.75]	[0.75, 1]

Table 3

l	$\mu_2(S_1^l)$	$\mu_2(S_2^l)$	$\mu_2(S_3^l)$
0	0	0.25	0.75
1	0	0.5625	0.4375
2	0.0017	0.5	0.4983
3	0.0017	0.5608	0.4375
4	0.0064	0.5	0.4936
5	0.0064	0.5561	0.4375

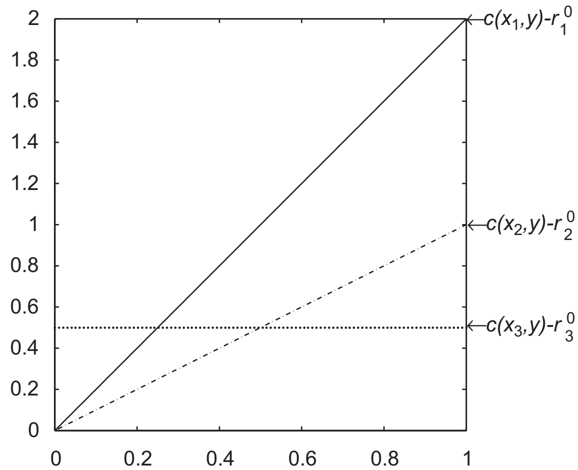

Fig. 1. 0th iteration.

At the 1st iteration, we translate $c(x_3, y)$. From the output results, we have $(r_1^1, r_2^1, r_3^1) = (0, 0, -\frac{1}{4})$, $S_1^1 = \{0\}$, $S_2^1 = [0, \frac{3}{4}]$, $S_3^1 = [\frac{3}{4}, 1]$, $\max_{1 \leqslant i \leqslant 3}\{\mu_2(S_i^1) - \mu_1(x_i)\} = \mu_2(S_2^1) - \mu_1(x_2)$. Thus at the 2nd iteration, we translate $c(x_2, y)$. At the 3rd iteration, we translate $c(x_3, y)$. At the 4th iteration, we translate $c(x_2, y)$. At the 5th iteration, we translate $c(x_3, y)$. We use the notation (*) to indicate which graph of $\{c(x_1, y), c(x_2, y), c(x_3, y)\}$ is to be translated.

At the 70th iteration, step 4 of the algorithm is satisfied. Hence, we stop the algorithm. We have $r_1{=}0$, $r_2{=}-0.25$, $r_3{=}-0.5$, $S_1{=}[0, \frac{1}{4}]$, $S_2{=}[\frac{1}{4}, \frac{3}{4}]$, $S_3 = [\frac{3}{4}, 1]$, $\mu_2(S_1) = \frac{1}{16} = \mu_1(x_1)$, $\mu_2(S_2) = \frac{1}{2} = \mu_1(x_2)$, $\mu_2(S_3) = \frac{7}{16} = \mu_1(x_3)$. Thus, there exists an optimal solution ρ for CTP with $\mathrm{supp}(\rho) = \bigcup_{i=1}^{3}\{x_i \times S_i\}$.

The graphs of the 0th iteration, 1st iteration, 2nd iteration, 3rd iteration, 4th iteration, and 70th are given in Figs. 1–6.

Example 4.2. Let x_1, x_2, x_3 be any three points in [0, 1]. We consider $c(x_1, y){=}2y$, if $0 \leqslant y \leqslant \frac{1}{2}$, $c(x_1, y){=}-2y+2$, if $\frac{1}{2} \leqslant y \leqslant 1$, and $c(x_2, y){=}y, 0 \leqslant y \leqslant 1$, $c(x_3, y){=}-y+1, 0 \leqslant y \leqslant 1$. Let $h(y){=}2y, 0 \leqslant y \leqslant 1$. Then $\mu_2(Y){=}\int_0^1 2y\,\mathrm{d}y{=}1$. Let $\mu_1(x_1) = \frac{1}{2}$, $\mu_1(x_2) = \frac{3}{16}$, $\mu_1(x_3) = \frac{5}{16}$.

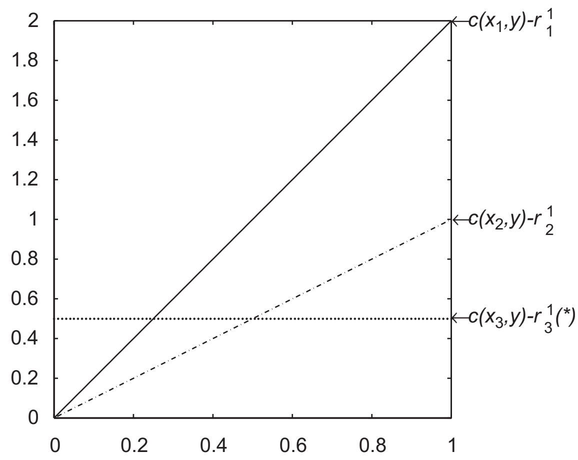

Fig. 2. 1st iteration.

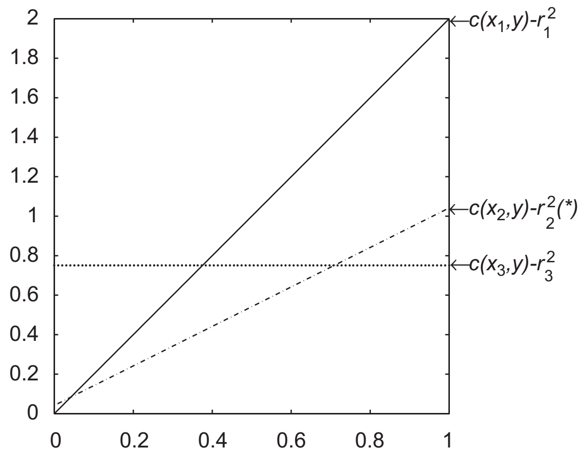

Fig. 3. 2nd iteration.

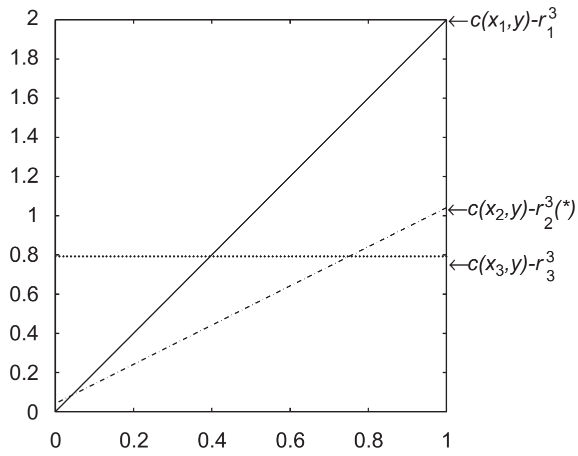

Fig. 4. 3rd iteration.

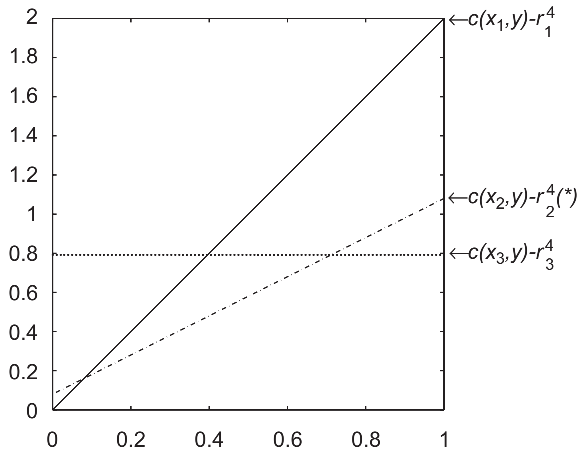

Fig. 5. 4th iteration.

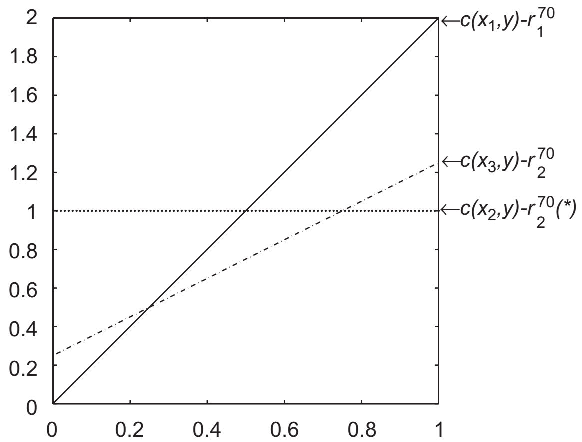

Fig. 6. 70th iteration.

At the 0th iteration, we let $(r_1^0, r_2^0, r_3^0) = (0, 0, 0)$. At the 20th iteration, step 4 of the algorithm is satisfied. Hence, we stop the algorithm. We have $r_1 = 0$, $r_2 = -0.25$, $r_3 = -0.25$, $S_1 = [0, 0.25] \cup [0.75, 1]$, $S_2 = [0.25, 0.5]$, $S_3 = [0.5, 0.75]$, $\mu_2(S_1) = \frac{1}{2} = \mu_1(x_1)$, $\mu_2(S_2) = \frac{3}{16} = \mu_1(x_2)$, $\mu_2(S_3) = \frac{5}{16} = \mu_1(x_3)$. Thus, there exists an optimal solution ρ for CTP with $\text{supp}(\rho) = \bigcup_{i=1}^{3} \{x_i \times S_i\}$. We use the notation (*) to indicate which graph of $\{c(x_1, y), c(x_2, y), c(x_3, y)\}$ is to be translated.

The graphs of the 0th iteration, 1st iteration, 2nd iteration, and 20th iteration are given in Figs. 7–10.

Example 4.3. Let x_1, x_2, x_3, x_4 be any four points in [0, 1]. We consider $c(x_1, y) = 3y$, $c(x_2, y) = 2y$, $c(x_3, y) = y$, $c(x_4, y) = \frac{1}{2}$, $0 \leqslant y \leqslant 1$. Let $h(y) = 2y$, $0 \leqslant y \leqslant 1$. Then $\mu_2(Y) = \int_0^1 2y \, dy = 1$. Let $\mu_1(x_1) = \frac{1}{16}$, $\mu_1(x_2) = \frac{3}{16}$, $\mu_1(x_3) = \frac{5}{16}$, $\mu_1(x_4) = \frac{7}{16}$.

At the 0th iteration, we let $(r_1^0, r_2^0, r_3^0, r_4^0) = (0, 0, 0, 0)$. At the 200th iteration, step 4 of the algorithm is satisfied. Hence, we stop the algorithm. We have $r_1 = 0$, $r_2 = -0.25$, $r_3 = -0.75$, $r_4 = -1$, $S_1 = [0, 0.25]$, $S_2 = [0.25, 0.5]$, $S_3 = [0.5, 0.75]$, $S_4 = [0.75, 1]$, $\mu_2(S_1) = \frac{1}{16} = \mu_1(x_1)$, $\mu_2(S_2) = \frac{3}{16} = \mu_1(x_2)$, $\mu_2(S_3) = \frac{5}{16} = \mu_1(x_3)$, $\mu_2(S_4) = \frac{7}{16} = \mu_1(x_4)$. Thus, there exists an optimal solution ρ for CTP with $\text{supp}(\rho) = \bigcup_{i=1}^{4} \{x_i \times S_i\}$.

We use the notation (*) to indicate which graph of $\{c(x_1, y), c(x_2, y), c(x_3, y), c(x_4, y)\}$ is to be translated.

The graphs of the 0th iteration, 1st iteration, 2nd iteration, and 200th iteration are given in Figs. 11–14.

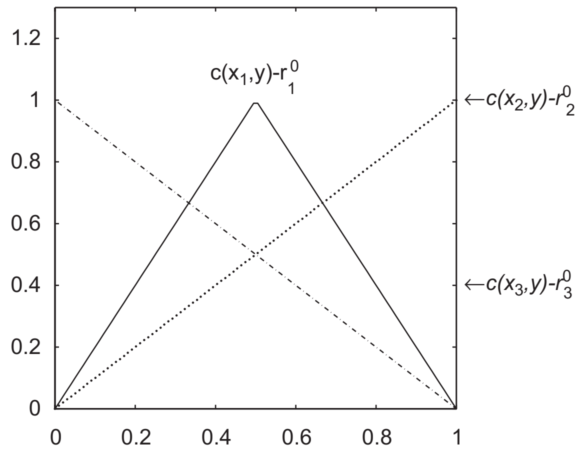

Fig. 7. 0th iteration.

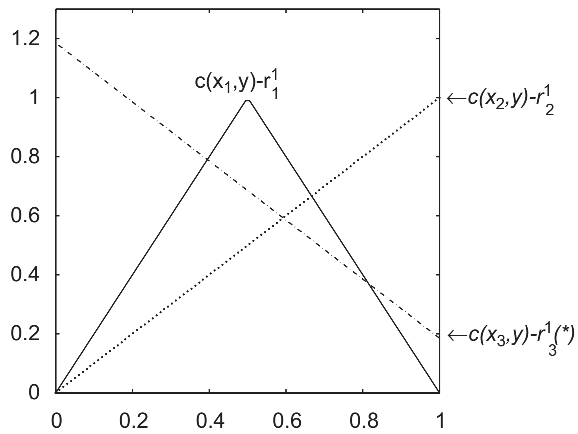

Fig. 8. 1st iteration.

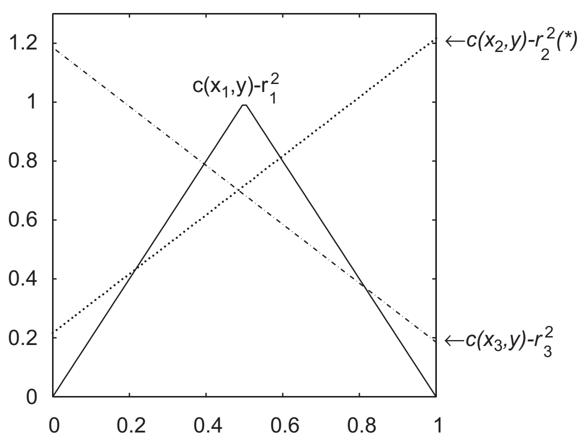

Fig. 9. 2nd iteration.

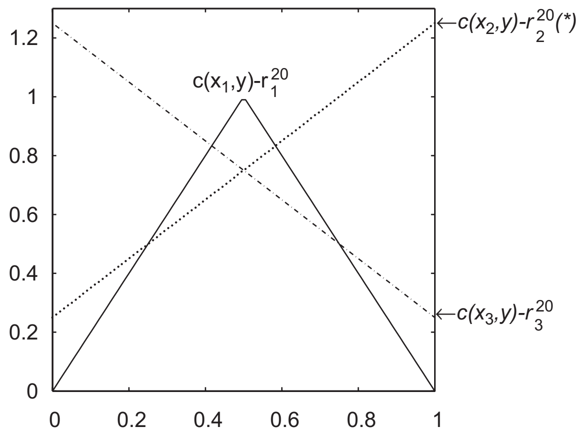

Fig. 10. 20th iteration.

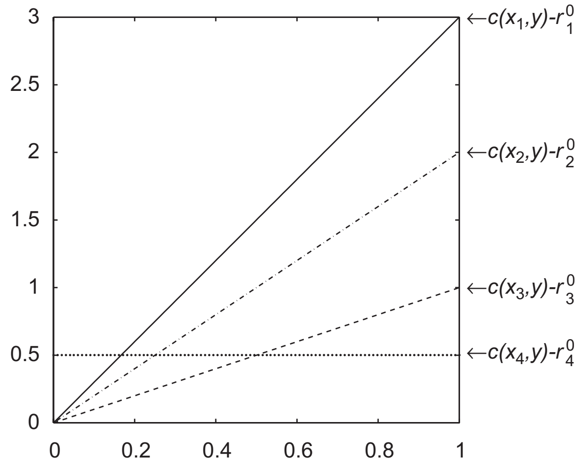

Fig. 11. 0th iteration.

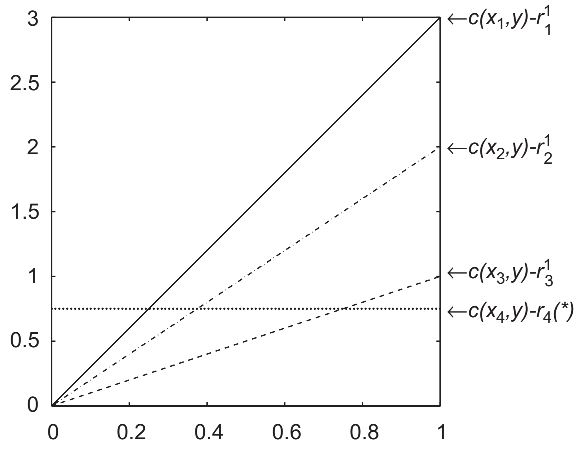

Fig. 12. 1st iteration.

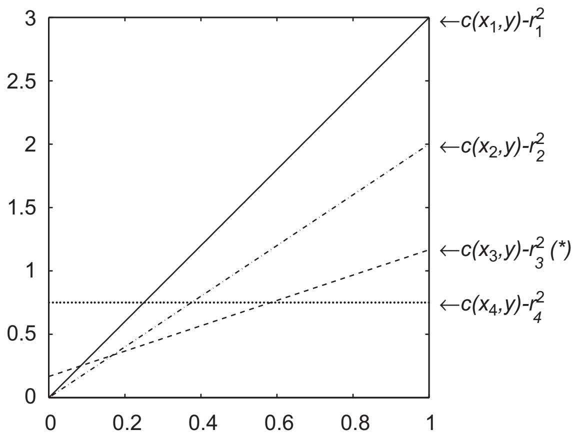

Fig. 13. 2nd iteration.

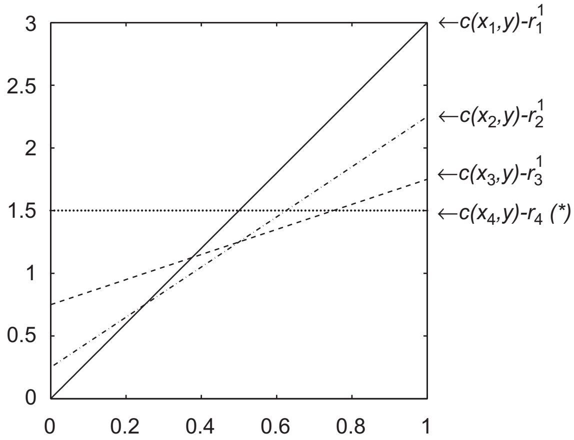

Fig. 14. 200th iteration.

Remarks. In Example 4.2, there exists at least one S_i^l which is a finite union of intervals of R. Example 4.3 is more complicated than Example 4.1. From the output results, it needs more iterations to obtain its optimal solution.

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Further Reading

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